

Paradoxes in Special Relativity

Dr. Robert A. Taylor,
University of Oxford, Department of Physics

In putting this lecture together I have drawn heavily on the work of Michael Weiss, which can be found as part of an excellent physics FAQ Web site at the following address: <http://hepweb.rl.ac.uk/ppUK/PhysFAQ/faq.html>. In particular, the discussion of the “Twin’s Paradox” is based largely on his work, and the entire appendix is his.

1. Einstein’s train paradox and simultaneity.

Lightning strikes the front and back ends of a moving train, leaving scorch marks on both track and train. An observer standing by the tracks halfway between them concludes that the strokes were simultaneous, as the flashes reach him at the same time. An observer riding in the middle of the train concludes that the strokes were not simultaneous. She says:

- (a) I am equidistant from the front and back scorch marks;
- (b) Light travels at c in my frame;
- (c) The flash arrived from the front of the train first;
- (d) Thus the flash must have left the front first, the front bolt fell before the back one – they are not simultaneous!

Therefore, two events that are SIMULTANEOUS in one inertial frame are NOT necessarily simultaneous in another.

2. Derivation of the Lorentz Transformations – included for completeness

The Lorentz Transformation is a LINEAR TRANSFORMATION. Consider two frames S and S' , S' is moving at speed v along x (neglect y and z for now) w.r.t. S .

What assumptions am I making? (These are Einstein’s postulates of special relativity)

- (a) The speed of light is the same in all inertial frames of reference.
- (b) The transformation depends only on the relative velocity of the two frames of reference and is symmetric w.r.t. them.
- (c) The transformation is a linear function, involving only first powers of the coordinates x, x and t, t' in S and S' .

Let:

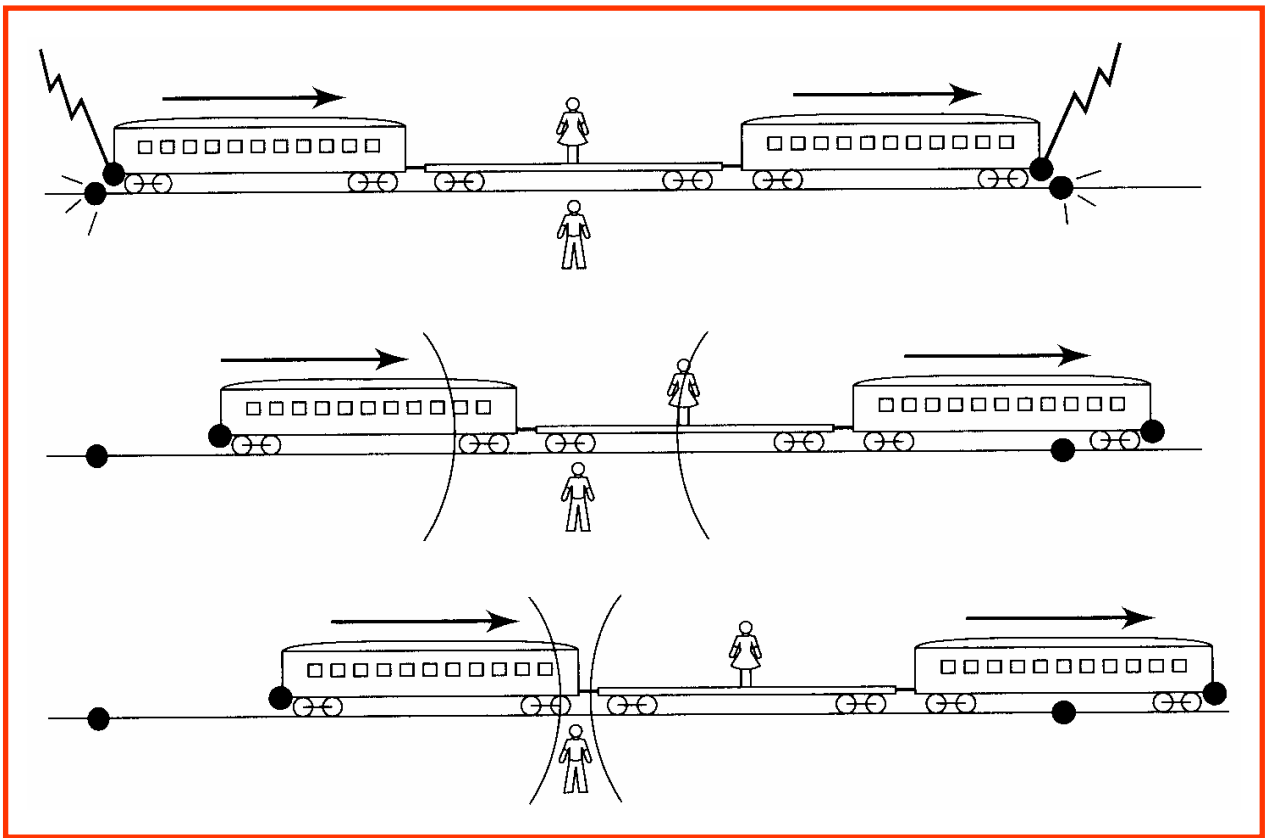
$$L(v) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{2}$$

$$x' = Ax + Bt \tag{3}$$

$$t' = Cx + Dt \tag{4}$$

The Train Paradox as posed by Einstein



Lightning strikes the front and back ends of a moving train, leaving scorch marks on both track and train. Observer standing by the tracks halfway between them concludes that the strokes were **simultaneous**, as the flashes reach him at the same time. Observer riding in the middle of the train concludes that the strokes were not simultaneous. She says:

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N.B. Two events that are **SIMULTANEOUS** in one inertial frame are **NOT** necessarily simultaneous in another

Coefficients B and C interrelate x and t and therefore must change sign with v .
Thus

$$L(-v) = \begin{pmatrix} +A & -B \\ -C & +D \end{pmatrix}$$

By condition (b)

$$L(v)L(-v) = 1$$

therefore

$$\begin{pmatrix} A^2 - BC & -AB + BD \\ CA - DC & -CB + D^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (5)$$

$$A = D \quad (6)$$

$$A^2 - BC = 1 \quad (7)$$

The origin of one frame of reference moves with velocity v in the other, so that if $x' = 0$, $x = vt$. Therefore

$$Avt + Bt = 0 \quad (8)$$

$$B = -Av \quad (9)$$

Now c is the same in both systems, so

$$\frac{x}{t} = c = \frac{x'}{t'} \quad (10)$$

$$c = \frac{Ax + Bt}{Cx + Dt} \quad (\text{eqn 3/eqn 4}) \quad (11)$$

$$c = \frac{Ac + B}{Cc + D} \quad (x = ct) \quad (12)$$

but $A = D$ and $B = -Av$, therefore

$$Ac - Av = Cc^2 + Ac \quad (13)$$

$$C = \frac{-Av}{c^2} \quad (14)$$

but $A^2 - BC = 1$ therefore

$$A^2 - \frac{A^2v^2}{c^2} = 1 \quad (15)$$

$$A^2 = \frac{1}{1 - v^2/c^2} \quad (16)$$

therefore

$$A = D = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

and

$$C = \frac{-v/c^2}{\sqrt{1 - v^2/c^2}} = \frac{-\gamma v}{c^2}$$

let $\beta = v/c$, then

$$L(v) = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & -c\beta\gamma \\ -\gamma\beta/c & \gamma \end{pmatrix}$$

Now replace t by ict and get

$$\begin{aligned} \mathcal{L}(v) &= \begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & \gamma \end{pmatrix} \\ \begin{pmatrix} x' \\ ict' \end{pmatrix} &= \mathcal{L}(v) \begin{pmatrix} x \\ ict \end{pmatrix} \end{aligned}$$

In this proof we already have $L(v)L(-v) = 1$ so the transforms are symmetric by definition! Thus the inverse transform is obtained by swapping primes and writing $-v$ for v . Exercise - check this!

3. What about y and z ? - Let

$$\begin{aligned} y &= \alpha y' \\ z &= \beta z' \end{aligned}$$

Then also we must have, by the relativity principle,

$$\begin{aligned} y' &= \alpha y \\ z' &= \beta z \end{aligned}$$

Then $yy' = \alpha\alpha y'y$, so

$$\begin{aligned} \alpha^2 &= 1 \\ \alpha &= \pm 1 \\ \beta &= \pm 1 \end{aligned}$$

Thus $y = y'$ and $z = z'$

4. What about the classical limit? - these transformations must still obey Newton's Laws in this limit. After all, Newton's mechanics has been very successful for many years! Take the limit $v \ll c$,

$$\begin{aligned} \gamma &= (1 - v^2/c^2)^{-\frac{1}{2}} \\ &\approx 1 + \frac{v^2}{2c^2} \end{aligned}$$

therefore to first order in v/c , $\gamma \rightarrow 1$ and

$$\begin{aligned} x' &= x - vt \\ t' &= t - \frac{vx}{c^2} \leftarrow 0 \\ t' &= t \end{aligned}$$

Thus we get the Galilean transformations back.

5. Now lets see why I chose to use ict earlier. Imagine in each frame a spherical wave spreading from the origin, they have the form:

$$\begin{aligned}x^2 + y^2 + z^2 - c^2 t^2 &= 0 \\x'^2 + y'^2 + z'^2 - c^2 t'^2 &= 0\end{aligned}$$

therefore for events on an expanding wavefront

$$\begin{aligned}x^2 - c^2 t^2 &= -(y^2 + z^2) & |x| &\leq ct \\x'^2 - c^2 t'^2 &= -(y'^2 + z'^2) & |x'| &\leq ct' \\y &= y' \\z &= z' \\x^2 - c^2 t^2 &= x'^2 - c^2 t'^2\end{aligned}$$

This is an example of invariance in special relativity, the square of the 4-vector is invariant under Lorentz transformation.

We can now write time as a space co-ordinate, we have four dimensions in SPACE-TIME given by (x_1, x_2, x_3, x_4) where $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$. The length of this vector in 4-space is fixed, so

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \text{constant} = x^2 + y^2 + z^2 - c^2 t^2$$

6. Now lets look at two very important consequences of the LTs, Lorentz-Fitzgerald contraction and time dilation. We'll start with time dilation.

Not only is the length of the 4-vector invariant, but so is interval between 2 events. Thus we have:

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2$$

Now, for two events in S' at the same place (e.g. clock ticks), we have $\Delta x = 0$. This is the clock rest frame and the time interval between 2 such events is the proper time denoted by $\Delta\tau$. So:

$$c^2 \Delta\tau^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Divide both sides by Δt^2 to get:

$$\left(\frac{\Delta\tau}{\Delta t}\right)^2 = 1 - v^2/c^2 \quad \left(\frac{d\mathbf{x}}{dt} = \mathbf{v}\right)$$

$$\Delta\tau = \Delta t \sqrt{1 - v^2/c^2} = \Delta t / \gamma$$

$$\Delta\tau < \Delta t$$

$\Delta\tau$ is the proper time - the time interval in the **REST FRAME** of the object, be it clock, particle, whatever! Thus we have:

$$\begin{aligned}\Delta t \text{ (for moving object)} &= \Delta\tau \times \gamma \\ \gamma &\geq 1 \\ \Delta t &\geq \Delta\tau \text{ (time dilation)}\end{aligned}$$

TIME DILATION

In the Lorentz Transformations the interval between 2 events is **INVARIANT**.

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2$$

Now for 2 events in S' at the same place (e.g. clock ticks), we have $\Delta x' = 0$. This is the clock **rest frame** and the time interval between 2 such events is the **proper time** denoted by $\Delta\tau$. So:

$$c^2 \Delta\tau^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Divide by Δt^2 then:

$$\left(\frac{\Delta\tau}{\Delta t} \right)^2 = 1 - v^2 / c^2 \quad \left(\frac{dx}{dt} \Big|_{S'} = v \right)$$

$$\therefore \Delta\tau = \Delta t \sqrt{1 - v^2 / c^2}$$

$$\therefore \Delta\tau < \Delta t$$

Therefore the time interval is **longer** than that in the rest frame - **TIME DILATION**

7. Lorentz-Fitzgerald Contraction

Consider a rigid rod of length l_0 at rest in frame S' , moving with velocity v w.r.t. to S . In S' use a metre rule to measure l_0 . In S , use light signals to measure x_1 and x_2 at the **SAME TIME**. From the Lorentz transformations we have:

$$x'_1 = \gamma(x_1 - vt) \quad x'_2 = \gamma(x_2 - vt)$$

so

$$\Delta x' = \gamma \Delta x$$

Since $\Delta x' = l_0$ we get $\Delta x = l_0/\gamma$ - Lorentz-Fitzgerald contraction. N.B. Measurements must be at the same time in S , whereas in S' you could measure one end of rod at t'_1 then have a cup of tea then measure other end at t'_2 . Doesn't matter since rod is at **REST** in S' .

l_0 is the **PROPER LENGTH** - measured in the rest frame of the rod.

8. The train in the tunnel paradox.

A train of proper length $2L = 500$ m approaches a tunnel of proper length $L = 250$ m. The train's speed u is such that $\gamma = 2$. An observer at rest with respect to the tunnel measures the train's length to be contracted by a factor of 2 to 250 m and expects the whole train to fit in the tunnel. An observer on the train knows that the length of the train is 500 m, and that the tunnel is contracted by a factor of 2 to 125 m. Thus the observer on the train argues that the train will not fit into the tunnel. Who is right?

We will solve this problem by considering two inertial frames, frame S , the rest frame of the tunnel and frame S' , the rest frame of the train. An observer in S would assign the position of the left side of the tunnel as $x = 0$ and the right side of the tunnel as $x = L$, while an observer in S' would assign the front of the train as $x' = 0$ and the rear of the train as $x' = -2L$. We check that, as measured in S , the train does indeed fit into the tunnel. Then we need to confirm that, as measured in S' , the train does **not** fit into the tunnel. In so doing we can explain this paradox.

We identify two events, marked by arrows on the diagram, and describe them in both frames. Events must be specified by both position and time. The first event is the entry of the train into the tunnel. The clocks in the two frames are set so that at $t = t' = 0$, the front of the train ($x' = 0$) coincides with the left side of the tunnel ($x = 0$), as measured by an observer in S at time $t = 0$ (see diagram (a)) and by an observer in S' at time $t' = 0$ (see diagram (b)). The second event is the alignment of the right end of the tunnel ($x = L$) and the front of the train ($x' = 0$), as measured by the observer in S (see diagram (c)) and the observer in S' (see diagram (d)), respectively.

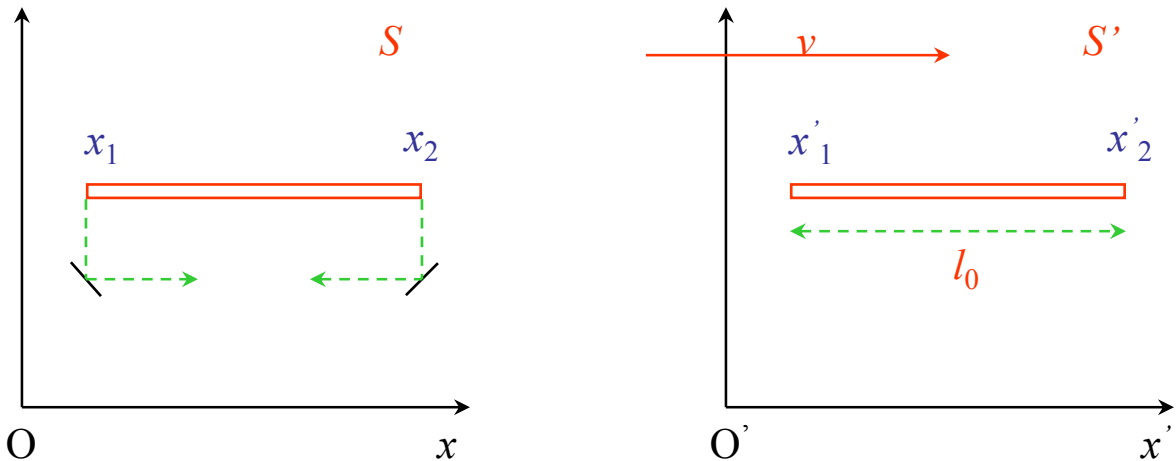
It follows from $x' = \gamma(x - ut)$ that, for the second event, $0 = \gamma(L - ut)$ so $t = L/u$. This is the time of the second event as seen by the observer in frame S . Now $t' = \gamma(t - ux/c^2)$ so

$$t' = \gamma \left(\frac{L}{u} - \frac{Lu}{c^2} \right) = \frac{L}{u} \sqrt{1 - \frac{u^2}{c^2}} = \frac{L}{u\gamma} = \frac{t}{\gamma}$$

LORENTZ CONTRACTION

Should be called Lorentz-Fitzgerald contraction

Consider a rigid rod of length l_0 at rest in frame S' , moving with velocity v with respect to S .



In S' use a metre rule to measure l_0 .

$$\Delta x' = x'_2 - x'_1 = l_0$$

In S , use light signals as shown to measure x_1 and x_2 at the SAME TIME

From the Lorentz transformations,

$$x'_1 = \gamma(x_1 - vt)$$

$$x'_2 = \gamma(x_2 - vt)$$

$$\Delta x' = \gamma \Delta x$$

$$\Delta x = l_0 / \gamma$$

This is the time of the second event as measured by the observer in S' . Where, according to the observer in S , is the rear of the train at $t = L/u$? We have

$$\begin{aligned}x' &= -2L = \gamma \left[x - u \left(\frac{L}{u} \right) \right] = \gamma(x - L) \\ \gamma &= 2 \\ x &= 0\end{aligned}$$

Thus at the time measured in S for the second event, the rear of the train is indeed at the tunnel entrance. As observed in S the train fits into the tunnel. By this we mean that the observer in S sees that the front is at the exit of the tunnel and the rear is at the entrance of the tunnel *at the same time*, which in this case is $t = L/u$.

Check that observer in S' does not see the train fit in the tunnel. That observer considers the train to be at rest and the tunnel to be moving towards the train. What is the location in S' of the tunnel entrance at the time of the event specified by the alignment of the front of the train and the tunnel exit—the value of x' when $x = 0$ at time $t' = L/\gamma u$? We have

$$\begin{aligned}x &= \gamma(x' + ut') \\ x &= 0 \\ x' &= -ut' = -L/\gamma = -L/2\end{aligned}$$

Thus, from the point of view of an observer in S' , only one-quarter of the train is in the tunnel when its front reaches the tunnel exit!

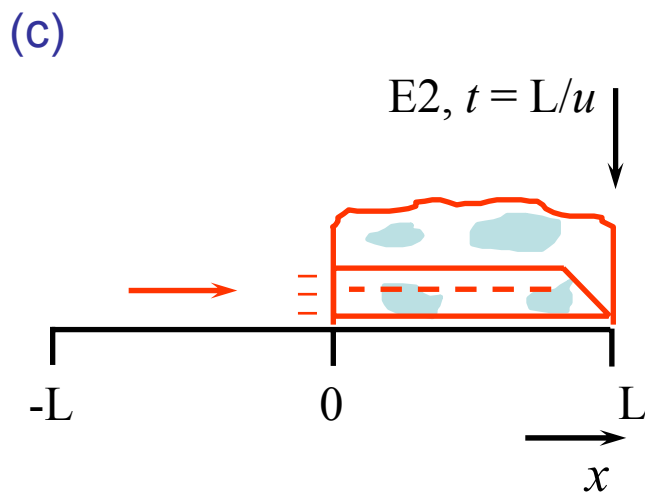
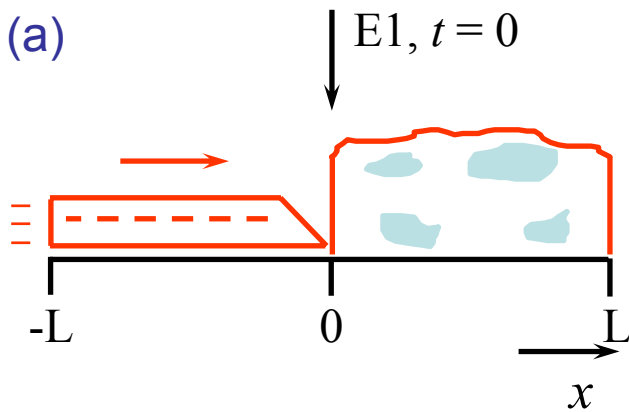
Let us make one last check and see in S' the time t' at which the rear of the train is at the tunnel entrance: What is t' when $x' = -2L$ and $x = 0$? We have

$$t' = \gamma \left[\frac{L}{u} - (0) \left(\frac{u}{c^2} \right) \right] = \frac{\gamma L}{u}$$

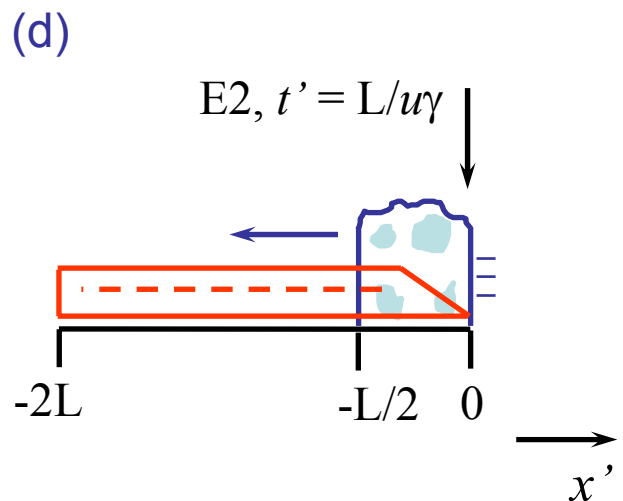
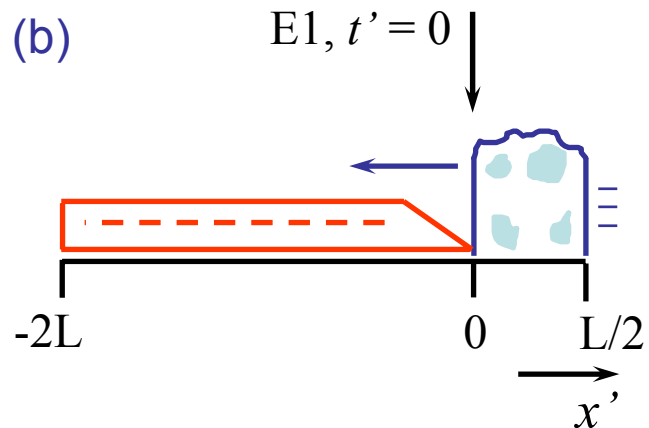
(not L/u as above).

9. This time is also different from the time $t' = L/u\gamma$ when the front of the train reaches the end of the tunnel. Thus the observer on the train in S' measures the end of the train to pass the tunnel entrance at a later time. The observer in S' will argue that the train's front passes the tunnel exit earlier than the rear passes the tunnel entrance. The differences between the interpretations of the two observers stem from their different notions of simultaneity. These differences allow both observers to be correct in their claims!
10. Assume terrorists build a wall at the end of the tunnel so that the train smashes into it and is brought to rest. Blood and guts, twisted metal and broken glass are everywhere of course, uugh! Let us assume, however, that the train is robust and made of the most rigid material we can imagine, so that when all the dust has cleared away it is essentially undamaged, with the front of the engine at rest against the wall. What will have appeared to have happened, as judged by our two observers? Since from our earlier arguments observers on the ground observe

Frame S (tunnel)



Frame S' (train)



The train's speed u is such that $\gamma = 2$.

This problem demonstrates once more the importance of the concept of **simultaneity** in relativity theory.

the train entirely in the tunnel for a while, presumably the end of the train does enter the tunnel. However, we also argued that to the train-based observer, the front of the train reached the front of the tunnel before the back enters the tunnel. So presumably in the new circumstances the back will never enter....???? Will the back have actually entered the tunnel or not? Where is the final resting place?

First, let us clear up what the final situation must be: since both tunnel and train end up at rest with respect to each other, the final position of the back end of the train must be L outside the tunnel! But did it go in and bounce out? Or did it approach the end from outside and then stop?

We return to our two events. Since there is a frame - the tunnel frame - in which the train is completely in the tunnel, the fact that there is a wall for the train to collide with cannot affect the fact that in that frame, the end of the train and tunnel actually coincided. And since it occurred in that frame, it occurred in everyone else's as well. In the S -frame, for example, it occurs at the same instant as the collision in the front.

Can we arrive at a contradiction by saying that in the train's frame we could make the opposite argument; that because the front of our rigid train came to rest before the back end had entered the tunnel, the back end can never do so? This sounds plausible, but does not stand up to closer scrutiny. The fact is that **the back end of the train has no means of finding out that the front has collided until it is already in the tunnel!** So it continues to move at its original speed until that information arrives, which cannot happen faster than the speed of light. By then, it is inside the tunnel, as I shall show below.

We are thus driven to the conclusion that the train really does disappear into the tunnel; that is, every frame of reference records the event where the end of the train passes the entrance of the tunnel. The train must therefore "bounce back" to its final resting place. As the front and back observer can only communicate at light speed, so can the molecules of the train. Thus we are forced to accept that the train really does compress; that observers on the train will notice that the molecules move closer together during the stopping process, eventually returning to their original relative positions (given that the train in fact is not a crumpled mess) after the collision. To illustrate this point consider the following:

What is the time difference between the front of the train hitting the entrance and the rear of the train entering the tunnel in frame S' ? From the discussion above this is just

$$\Delta t'_1 = \frac{\gamma L}{u} - \frac{L}{u\gamma}$$

but here $\gamma = 2$. So we have

$$\Delta t'_1 = \frac{2L}{u} - \frac{L}{2u} = \frac{L}{u} \left[2 - \frac{1}{2} \right]$$

therefore

$$\Delta t'_1 = \frac{3L}{2u}$$

Now assume an observer at the front of the train sends a light signal to the rear just as the front collides with the wall. This signal travels $2L$ in the rest frame of

the train, and so takes a time

$$\Delta t'_2 = \frac{2L}{c}$$

to arrive at the rear of the train. Now for $\gamma = 2$ we need to know u . Thus

$$\begin{aligned} 2 &= \frac{1}{(1 - u^2/c^2)^{1/2}} \\ 4 &= \frac{1}{(1 - u^2/c^2)} \\ 1 - u^2/c^2 &= \frac{1}{4} \\ u^2/c^2 &= \frac{3}{4} \\ u &= 0.866c \\ \Delta t'_1 &= (1.5/0.866c) L = 1.73L/c \\ \Delta t'_2 &= \frac{2L}{c} \\ \Delta t'_2 &> \Delta t'_1 \end{aligned}$$

therefore the rear of the train is in the tunnel before any signal from the front warning of a collision can reach it. The train must fit in the tunnel! After they all come to rest the train is a length L **out** of the tunnel. The conclusion is the relativity tells us that no material object is perfectly rigid, the train must compress!

11. The Twins Paradox

Our story stars two twins, Stella and Terence. Terence sits at home on Earth. Stella flies off in a spaceship at nearly the speed of light, turns around after a while, thrusters blazing, and returns. (So Terence is the terrestrial sort; Stella sets her sights on the stars.)

When our heroes meet again, what do they find? Did time slow down for Stella, making her years younger than her home-bound brother? Or can Stella declare that the Earth did the travelling, so Terence is the younger?

Perhaps we can make short work of the “travelling Earth” argument. Special relativity does not declare that all frames of reference are equivalent, only so-called inertial frames. Stella’s frame is not inertial while she is accelerating. And this is observationally detectable: Stella had to fire her thrusters midway through her trip; Terence did nothing of the sort. The Ming vase she had borrowed from Terence fell over and cracked. She struggled to maintain her balance, like the crew of Star Trek. In short, she felt the acceleration, while Terence felt nothing.

Whew! One short paragraph, and we’ve polished off the twin paradox. Is that really all there is to it? Well, not quite. There’s nothing wrong with what we’ve said so far, but we’ve left out a lot. There are reasons for the popular confusion.

For one thing, we've been rather unfair to Stella. We've said why she can't simply adopt Terence's viewpoint, but we haven't said how things look from her perspective. It seems strange that Terence could age several years just because Stella engages her thrusters. The Time Gap and Distance Dependence Objections put a sharper edge on this uneasy feeling.

Let's lay out a "standard version" of the paradox in detail, and settle on some terminology. We'll get rid of Stella's acceleration at the start and end of the trip. Stella flashes past Terence in her spaceship both times, coasting along.

Here's the itinerary *according to Terence*:

Start Event

Stella flashes past. Clocks are synchronized to 0.

Outbound Leg

Stella coasts along at (say) nearly 99 percent of light-speed. At 99 percent, the time dilation factor is a bit over 7, so let's say the speed is just a shade under 99 percent and the time dilation factor is 7. Let's say this part of the trip takes 7 years (according to Terence, of course).

Turnaround

Stella fires her thrusters for, say, 1 day, until she is coasting back towards Earth at nearly 99 percent light-speed. (Stella is the hardy sort.) Some variations on the paradox call for an instantaneous Turnaround; we'll call that the Turnaround Event.

Inbound Leg

Stella coasts back for 7 years at 99 percent light-speed.

Return Event

Stella flashes past Terence in the other direction, and they compare clocks, or grey hairs, or any other sign of elapsed time.

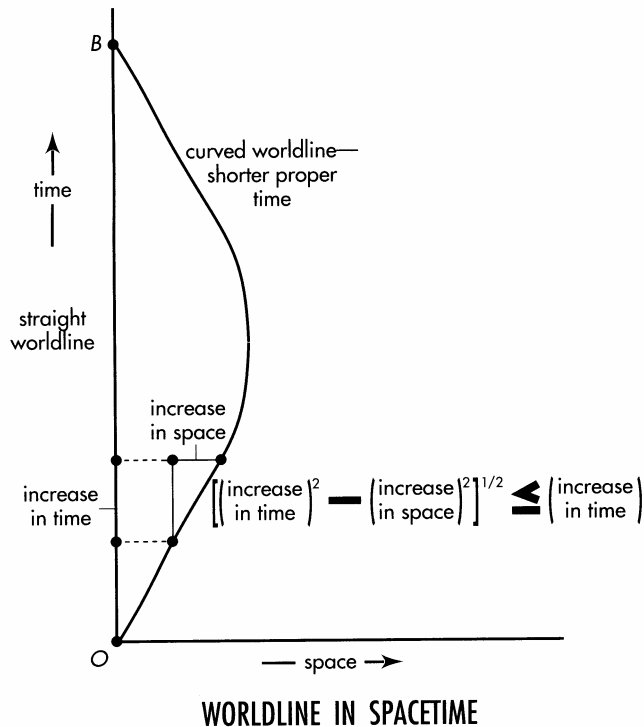
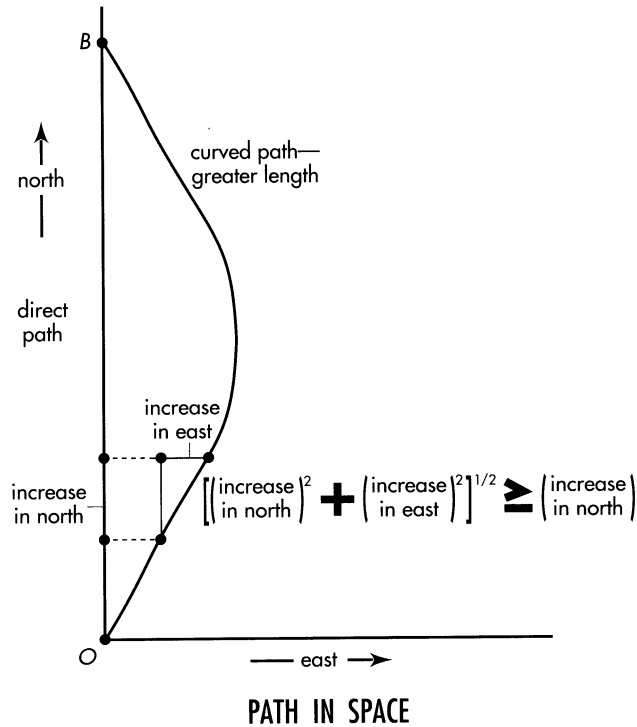
According to Terence, 14 years and a day have elapsed between the Start and Return Events; Stella's clock however reads just a shade over 2 years.

How much over? Well, Terence says the Turnaround took a day. Stella's speed was changing throughout the Turnaround, and so her time dilation factor was changing, varying between 1 and 7. So Stella's measurement of the Turnaround Time will be something between 1 day and one-seventh of a day. If you work it out, it turns out to be a bit over 15 hours.

The Twins Paradox: The Doppler Shift Explanation

Let us focus on what Stella and Terence actually **see** with their own eyes. (Just to emphasize that we're talking about direct observation here, I'll put the verb "see" and its brothers in bold throughout this section.) To make things interesting, we'll equip them with unbelievably powerful telescopes, so each twin can **watch** the other's clock throughout the trip. If each twin **saw** the other clock run slow throughout the trip, then we *would* have a contradiction. But this is *not* what they see.

Principle of Maximal Ageing



Path in space: In Euclidean geometry the curved path has greater length

Worldline in spacetime: In Lorentz geometry the curved worldline is traversed in the shorter proper time

Just in case it's too hard to read the clock hands through the telescope, we'll add a flash unit to each clock, set to flash once a second. You might guess at first that Terence **sees** Stella's clock flashing once every 7 seconds (with the time dilation factor we've chosen) and vice versa. Not so! On the Outbound Leg, Terence **sees** a flash rate of approximately one flash per 14 seconds; on the Inbound Leg, he **sees** her clock going at about 14 flashes per second. That is, he **sees** it running *fast*! Stella **sees** the same behavior in Terence's clock.

What gives? Well, the section title gave it away: just replace the words "flashes per second" with "cycles per second", and you'll recognize the familiar Doppler shift at work. The regular pulses are red-shifted to lower frequencies during the Outbound Leg, and blue-shifted to higher frequencies during the Inbound Leg. (I invite you to consider laser-based clocks instead of flash units, for added technical effect.)

The Doppler shift factors I gave (1/14 and 14/1) come from the relativistic Doppler formula. The relativistic formula takes into account both the "delay through distance" effect of the non-relativistic formula, and the relativistic time dilation. This is

$$\nu' = \nu_0 \gamma (1 \mp \beta)$$

In other words, Terence *computes* that Stella's clock is really running slow by a factor of about 7 the whole time, but he **sees** it running fast during the Inbound Leg because each flash has a shorter distance to travel. And Stella computes the same for Terence.

All well and good, but this discussion at first just seems to sharpen the paradox! Stella **sees** what Terence **sees**: a slow clock on the Outbound Leg, a fast clock on the Inbound Leg. Whence comes the asymmetry between Stella and Terence?

Answer: in the length of the Inbound and Outbound Legs, as **seen**. For Stella, each Leg takes about a year. Terence maintains that Stella's Turnaround takes place at year 7 at a distance of nearly 7 light-years, so he won't **see** it until nearly year 14. Terence **sees** a long Outbound Leg and a very short Inbound Leg.

So there's the fundamental asymmetry: the switch from red-shift to blue-shift occurs at *Stella's* Turnaround. Stella **sees** Terence's telescopic image age slowly on her Outbound Leg, but the image more than makes up for its dawdling on the Inbound Leg. Terence **sees** Stella's image off to a slow start too, but here the image's final burst of rapid aging comes too late to win the race.

The Twins Paradox: The Spacetime Diagram Explanation

Minkowski said, "Henceforth Space by itself, and Time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." Minkowski recast Einstein's version of Special Relativity on a new stage, Minkowski spacetime. The Twin Paradox has a very simple resolution in this framework. The crucial concept is the proper time of a moving body (see section 6).

Recall from above that for proper time:

$$c^2 \Delta \tau^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

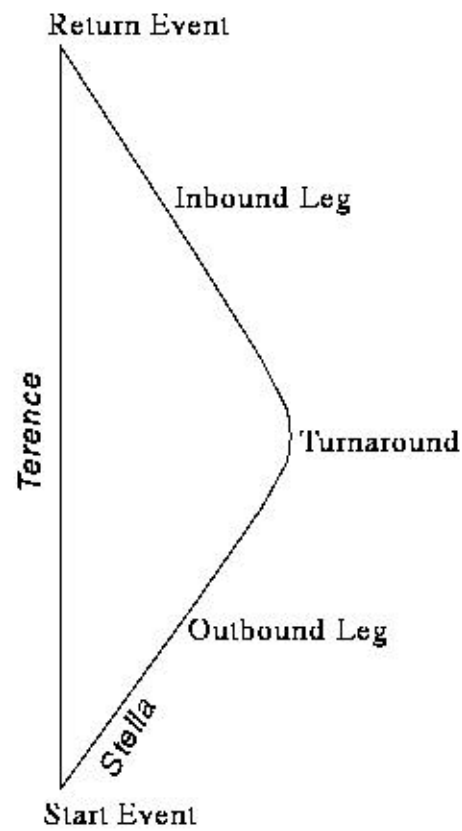


Figure 1: The Twin's Worldlines

Stella sends
pulses to Terence

Terence sends
pulses to Stella

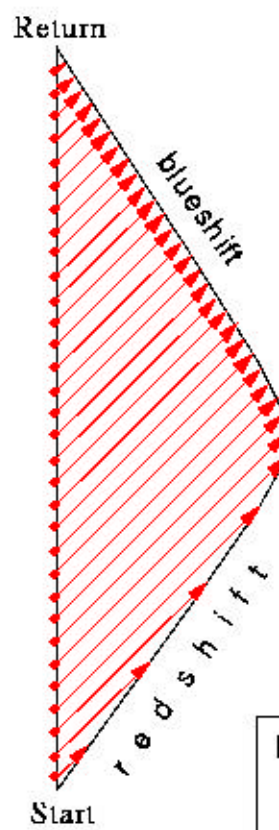
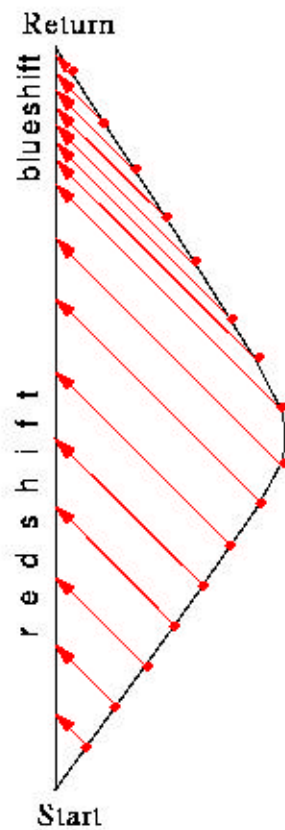


Figure 2: The Doppler Shift Explanation

Using the fact that spacetime can be represented by four co-ordinates, the fourth being ct , we can draw a diagram with x as the vertical axis and ct as the horizontal axis. This is known as a Minkowski spacetime diagram. It has the peculiar property that for any path, not along the vertical axis, the time taken will always be less than that for the path along the vertical axis, since this corresponds to the proper time with $\Delta x = \Delta y = \Delta z = 0$. All other paths have a positive space contribution to the interval, making the time part necessarily smaller in order to preserve the invariance. This is illustrated in the accompanying figures, and is known as the Principle of Maximal Ageing. Thus if we evaluate a path integral along two different paths, we get two different results, the largest being always along the vertical or space axis. Not so different in spirit from picking two points in ordinary Euclidean space, and then evaluating the arc-length integral along two different paths connecting them. It's not just where you're going, it's how you get there.

Applying this to our particular problem we come to the conclusion that the value of Stella's integral is less than that of Terence's integral, *i.e.* her elapsed proper time is less than Terence's; *i.e.* she ages less.

The Twins Paradox: The Time Gap Objection

Make the Turnaround instantaneous. Relativity puts an upper on speed, but no upper limit on acceleration. An instantaneous Turnaround Event is the limiting case of shorter and shorter Turnarounds, and so the theory should handle it.

During the Outbound Leg, Terence ages less than two months, according to Stella. (12 Stella-months/time dilation factor of 7.) During the Inbound Leg, Terence also ages less than two months, according to Stella, by the same computation. The Turnaround Event is instantaneous. Total Terence ageing: less than 4 months, it would seem. Yet Terence is supposed to be over 14 years older when Stella returns! Where did the missing time go?

The Doppler Shift Explanation makes short work of this. Stella sees (through her telescope) Terence age hardly at all during her Outbound Leg, and nearly the full 14 years during her Inbound Leg. No gap.

Of course, she calculates something different, taking into account Doppler shifts and the finite speed of light. These calculations must be based on reference frames. In fact, for Stella to get two months for the Outbound Leg and two months for the Inbound Leg (as in the paragraph above), she has to switch inertial reference frames midway through her journey.

Now different inertial reference frames have different notions of simultaneity. The Outbound reference frame says: "At the same time that Stella makes her Turnaround, Terence's clock reads about two months." The Inbound reference frame says: "At the same time that Stella makes her Turnaround, Terence's clock reads about 13 years and 10 months." The apparent "gap" is just an accounting error, caused by switching from one frame to another.

The Spacetime Explanation offers the same elucidation, graphically expressed.

The Twins Paradox: The Distance Dependence Objection

With our "standard example" (see above), Stella's accounting of Terence's ageing runs like this: one-seventh of a year on the Outbound Leg, one-seventh of a

year on the Inbound Leg, and the rest — 14 years minus two-sevenths — during the Turnaround. You may recall she does the Turnaround in a day, according to Terence, or about 15 hours by her own clock. (Let's just say 15, and hang the minutes; the exact figure won't matter.) Say Stella takes a longer journey, spending 2 years on both the Inbound and Outbound Legs, for a total of 4 years of her time, or 28 years according to Terence. But she still takes the same 15 hours for the Turnaround.

So when Stella and Terence have their joyous reunion, Terence is 28 years older (plus a day). This time Stella's accounting of Terence's ageing runs like so: Terence aged two-sevenths years on the Outbound Leg, ditto for the Inbound Leg, and so Terence must have aged over 27 years during the Turnaround.

Summing up: according to Stella, Terence ages around 13 years 7 months on the Turnaround or the shorter trip, but over 27 years on the Turnaround for the longer trip. Yet Stella says the two Turnarounds took the same time. And Terence agrees.

The resolutions are similar to those given for the Time Gap Objection. How much Terence ages during the Turnaround is not something you can directly observe, according to special relativity. The Doppler Shift Explanation focuses on what Terence and Stella actually see through their telescopes, which avoids the difficulty. Stella's accounting is just that: accounting, dependent on particular reference frames, and in particular on switching from one inertial reference frame to another. No wonder that accounting tricks can produce surprising results. See the Time Gap Objection for more details.

12. Space travel — is it possible?

99 light years from earth is Canopus. We are asked to go there, photograph it and return home. But that's impossible we say, only 40 years or so is left of our lives and the round trip is 198 years! Assume that our ship can travel at a speed of $\frac{99}{101}c$ relative to the Earth. We can therefore travel 99 light years in 101 years in Earth's frame ($v = 0.9802c$). The whole trip will therefore take 202 years in Earth's frame. What about rocket time (rocket's inertial frame). Rocket time is proper time

$$\begin{aligned}\Delta\tau &= \Delta t/\gamma \\ \Delta\tau &= 101/\gamma \\ \&\ \gamma &= \frac{1}{(1 - (0.9802)^2)^{1/2}} = 5.05 \\ \Delta\tau &= 20 \text{ years!}\end{aligned}$$

Or we can use the invariance of the interval

$$(\Delta x_{\text{rkt}})^2 - (c\Delta t_{\text{rkt}})^2 = (\Delta x_{\text{earth}})^2 - (c\Delta t_{\text{earth}})^2$$

but $\Delta x_{\text{rkt}} = 0$, therefore

$$\begin{aligned}-\Delta t_{\text{rkt}}^2 &= \left(\frac{\Delta x_{\text{earth}}}{c}\right)^2 - \Delta t_{\text{earth}}^2 \\ &= 99^2 - 101^2 = -20^2 \text{ years}^2 \\ \Delta t_{\text{rkt}} &= 20 \text{ years}\end{aligned}$$

thus space travel is possible on paper.

13. Olbers's Paradox

Heinrich Olbers (1926): If the Universe is uniformly filled with stars, then no matter which direction you look, your line of sight will eventually intersect a star (or other bright object).

Therefore, why is the night sky black?

Proof:

Average Luminosity per star: L Number density of stars: n stars / volume Brightness of star at distance r : $B = L/4\pi r^2$

Examine shell of stars at distance r . Total brightness from shell with thickness dr is:

$$\begin{aligned} B_{\text{shell}} &= [\text{volume of shell}] \times [\text{density of stars}] \times [\text{brightness per star}] \\ &= 4\pi r^2 dr \times n \times L/4\pi r^2 = nL dr. \end{aligned}$$

This is independent of distance, and as we add more shells increases without limit! Even with absorption etc. we still expect the sky to be as brilliant as the surface of a star!

Resolution:

- (a) The universe is of finite age - light from stars more distant than ct_0 hasnt reached us yet.
- (b) The universe is expanding - Doppler shift reduces energy of photons, fluxes drop faster than $1/r^2$.

In fact explanation (b) has only a factor of 2 influence on the brightness. Olbers's paradox needs a universe of finite age!

(ref. The Astrophysical Journal, **317**, 601-606, 1987).

APPENDIX for the masochistic

The Twins Paradox: The “General Relativity” Explanation

Some background

This explanation relies on a couple of assertions that are true, according to general relativity. (We postpone asking what special relativity has to say about these assertions.)

1. Free choice of reference frames: You can describe the physics of a situation in pretty much any reference frame you like, but some frames demand the introduction of force fields that don't show up in other frames. You can call these “pseudo-force fields”, or even “gravitational fields”.
2. Uniform “gravitational” time dilation: Say you have two identically constructed clocks. One is deep down in a uniform “gravitational” potential well (or “pseudo-potential” well, if you prefer); the other is higher up. If the two clocks compare rates by sending light signals back and forth, then both will agree that the lower clock runs slower than the higher clock.

I've phrased these statements as carefully as I know how, short of resorting to formal mathematics. The short and sloppy versions say, “You can use accelerated frames of reference, so long as you throw in some pseudo-force fields”; and “Time runs slower as you descend into the potential well of a uniform pseudo-force field.”

Older books called our first assertion the General Principle of Relativity, but that term has fallen into disuse. It may remind you of the Principle of Equivalence, but that's really something different. The short and sloppy version of that says you can't distinguish a gravitational field due to matter from a pseudo-force field. We won't need the Principle of Equivalence here, so I won't bother with a more careful statement. Even so, you can probably see the connections: without Equivalence, uniform “gravitational” time dilation has nothing whatever to do with gravity. (Call it “pseudo-force time dilation” instead.)

OK, now for the twins paradox

Our usual version, that is. We'll pick a frame of reference in which Stella is at rest the whole time! When she ignites her thrusters for the Turnaround, she is forced to assume that a uniform “gravitational” field suddenly permeates the universe; the field exactly cancels the force of her thrusters, so she stays motionless.

Not so Terence. The field causes him to accelerate, but he feels nothing new since he's in free-fall (or rather the Earth as a whole is). There's an enormous potential difference between him and Stella: remember, he's light-years from Stella, in a uniform “gravitational” field! Stella's at the bottom of the well, he's at the top (or they would be, if the well weren't bottomless and topless). So by uniform “gravitational” time dilation, he ages years during Stella's Turnaround.

Short and sweet, once you have the background! As an added bonus, the general relativity explanation makes short work of Time Gap and Distance Dependence Objections. The Time Gap Objection invites us to consider the limit of an instantaneous Turnaround. But in that limit, the “gravitational” field becomes infinitely strong, and so does the time dilation. So Terence ages years in an instant – physically unrealistic, but so is instantaneous Turnaround.

The Distance Dependence Objection finds it odd that Terence's Turnaround ageing should depend on how far he is from Stella when it happens, and not just on Stella's

measurement of the Turnaround time. No mystery: uniform “gravitational” time dilation depends on the “gravitational” potential difference, which depends on the distance.

You may be bothered by the Big Coincidence: how come the uniform “gravitational” field happens to spring up just as Stella engages her thrusters? You might as well ask children on a merry-go-round why centrifugal force suddenly appears when the carnival operator cranks up the engine. There’s a reason such forces have had to endure the derisive prefix “pseudo” in so many books.

You may find uniform “gravitational” time dilation, the second assertion, a mite too convenient. Where did it come from? Is it just a fudge factor that Einstein introduced to resolve the twin paradox? Not at all. Einstein gave a couple of derivations for it, having nothing to do with the twin paradox. These arguments don’t need the Principle of Equivalence. I won’t repeat Einstein’s arguments (chase down some of the references if you’re curious), but I do have a bit more to say about this effect in the section titled Too Many Explanations.

Gravitation time dilation without scare quotes (i.e., fields due to matter) is a different story. These fields are never uniform, and the derivations just mentioned don’t work. The essence of Einstein’s first insight into General Relativity was this: (a) you can derive time dilation for uniform “gravitational” fields; (b) the Principle of Equivalence then implies time dilation for gravitational fields (no scare-quotes). A stunning achievement, but irrelevant to the twin paradox.

What is General Relativity?

Einstein worked on incorporating gravitation into relativity theory from 1907 to 1915; by 1915, general relativity had assumed pretty much its modern form. (Oh, the mathematicians found some spots to apply polish and gold-plating, but the conceptual foundations remain the same.) If you asked him to list the crucial features of General Relativity in 1907, and again in 1915, you’d probably get very different lists. Certainly modern physicists have a different list from Einstein’s 1907 list.

Here’s my version of Einstein’s 1907 list (without worrying too much about the fine points):

1. General Principle of Relativity

All motion is relative, not just uniform motion. You will have to include so-called pseudo-forces, however (like centrifugal force or Coriolis force).

2. Principle of Equivalence

Gravity is not essentially different from any pseudo-force.

The General Principle of Relativity plays a key role in the general relativity explanation of the twin paradox. And this principle gave General Relativity its name. So there’s certainly historical justification for the term “general relativity explanation”. Even in 1916, Einstein continued to single out the General Principle of Relativity as a central feature of the new theory. (See for example the first three sections of his 1916 paper, “The Foundation of the General Theory of Relativity”, or his popular exposition Relativity.) Here’s the modern physicist’s list (again, not sweating the fine points):

1. Spacetime Structure

Spacetime is a 4-dimensional Riemannian manifold. If you want to study it with coordinates, you may use any smooth set of charts (aka local coordinate systems). (This free choice is what has become of the General Principle of Relativity.)

2. Principle of Equivalence

The metric of spacetime induces a Minkowski metric on the tangent spaces. In other words, to a first-order approximation, a small patch of spacetime looks like a small patch of Minkowski spacetime. Freely falling bodies follow geodesics.

3. Gravitation = Curvature

A gravitational field due to matter exhibits itself as curvature in spacetime. In other words, once we subtract off the first-order effects by using a free-falling frame of reference, the remaining second-order effects betray the presence of a (true) gravitational field.

The third feature finds its precise mathematical expression in the Einstein field equations. This feature looms so large in the final formulation of general relativity, that most physicists reserve the term “gravitational field” for the fields produced by matter. The phrases “flat portion of spacetime”, and “spacetime without gravitational fields” are synonymous in modern parlance. “Special relativity” and “flat spacetime” are also synonymous, or nearly so; one can quibble over whether flat spacetime with a non-trivial topology (for example, cylindrical spacetime) counts as “special relativity”. Incidentally, the “modern” usage appeared quite early. Eddington’s book *The Mathematical Theory of Relativity* (1922) defines Special Relativity as the theory of flat spacetime.

So modern usage demotes the uniform “gravitational” field back to its old status as a pseudo-field, with all the pejorative connotations of the prefix “pseudo”. And the hallmark of a truly general relativity problem (i.e., not special relativity) is that spacetime is not flat. By contrast, the free choice of charts – the modern form of the General Principle of Relativity – doesn’t pack much of a punch. You can use curvilinear coordinates in flat spacetime. (If you use polar coordinates in plane geometry, have you suddenly departed the kingdom of Euclid?)

The usual version of the twin paradox qualifies as a pure special relativity problem, by modern standards. Spacetime is ordinary flat Minkowski spacetime. Stella’s frame of reference is just a curvilinear coordinate system.

The Spacetime Diagram Explanation is closer to the spirit of general relativity (vintage 1916) than the so-called “general relativity” explanation. Spacetime, geodesics, and the invariant interval: that’s the core of general relativity.

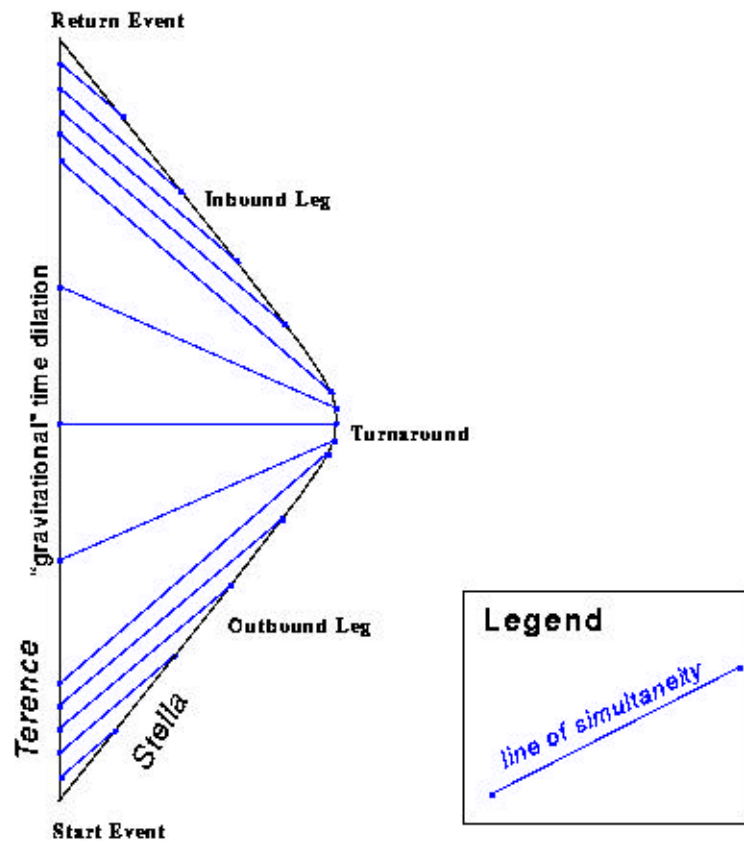


Figure 3: The "GR" explanation

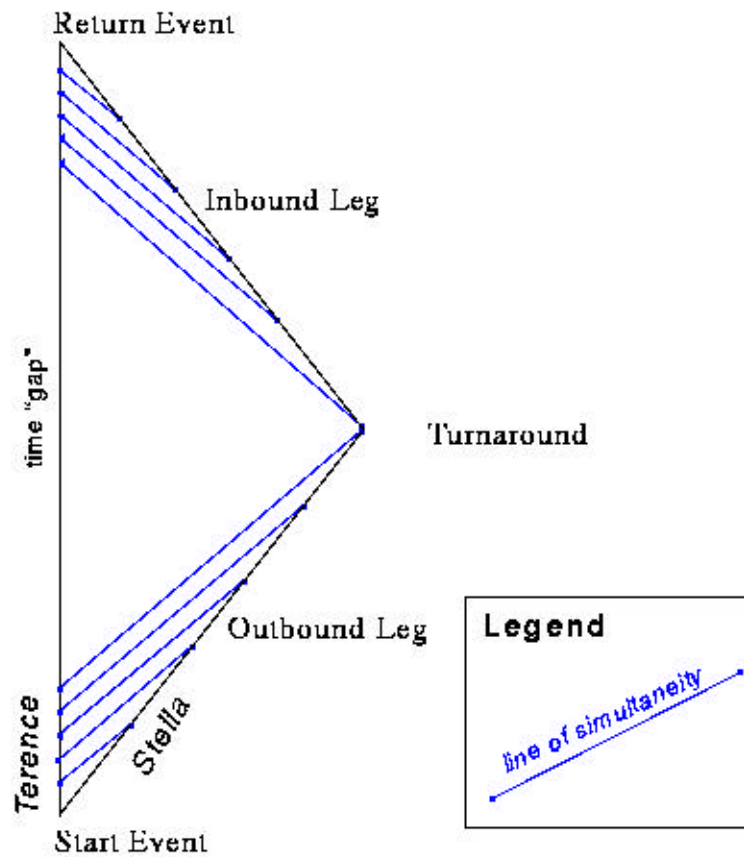


Figure 4: The Time Gap Objection